doi:10.1006/jabr.2001.9050, available online at http://www.idealibrary.com on IDEAL

On the Grauert–Riemenschneider Vanishing Theorem

Juan Elias[1]

Departament d'Àlgebra I Geometria, Facultat de Matemàtiques, Universitat de Barcelona, Gran Via 585, 08007 Barcelona, Spain
E-mail: elias@mat.ub.es

Communicated by Craig Huneke

Received May 30, 2001

INTRODUCTION

Let R be a Cohen–Macaulay local ring essentially of finite type over $\mathbb{C}$ and let $X = \operatorname{Proj}(\mathscr{R}(I))$ be the blowing up of $\operatorname{Spec}(R)$ along the closed subscheme defined by an ideal I of R. Let Y be the closed fiber of $\pi\colon X \to \operatorname{Spec}(R)$. Sancho de Salas proved that if X is a Cohen–Macaulay scheme then

SdS-1. There exists an integer n_0 such that the associated graded ring $\operatorname{gr}_{I^n}(R)$ is Cohen–Macaulay for all $n \geq n_0$ if and only if $H^i_Y(X, \mathscr{O}_X) = 0$ for $i < \dim(R)$.

In the proof of this result Sancho de Salas introduced a new exact sequence relating some local cohomology groups of R and sheaf cohomology groups of X. We use here an algebraic version of this sequence due to Karen Smith (see [12, p. 150]).

From **SdS-1** Sancho de Salas deduces the following version of the Grauert–Riemenschneider vanishing theorem ([5, Satz 2, 3] and [6, Proposition 2.2]):

SdS-2. If X is non-singular then $\operatorname{gr}_{I^n}(R)$ is Cohen–Macaulay for all large values of n.

Lipman in [12, Theorem 4.3] proved that **SdS-1** holds for all Cohen–Macaulay local ring R without the assumption that it is essentially of finite type over $\mathbb{C}$.

[1] Partially supported by a grant from the Dirección General de Investigación Cientifica y Ténica (PB97-0893).

0021-8693/02 $35.00

Cutkosky [3, Part III] and Huckaba and Huneke [8, Theorem 3.12] give examples showing that the **SdS-2** result does not holds for a d-dimensional Cohen–Macaulay ring R, with $d \geq 3$ and such that X is a normal scheme instead of a non-singular scheme.

Huneke proved in [11, Exercises 5.12 and 5.13] an algebraic version of the Grauert–Riemenschneider vanishing theorem (see also [8, Corollary 3.8]):

aGR. Let R be a two-dimensional Cohen–Macaulay local ring, and let I be a normal $\mathbf{m}$-primary ideal. Then $\mathrm{gr}_{I^n}(R)$ is Cohen–Macaulay for all large values of n.

This result has been generalized by Huckaba and Marley in [10, Corollary 3.9].

Valla in [17] proved that if R is Cohen–Macaulay and I is a complete intersection ideal then $\mathrm{gr}_{I^n}(R)$ and $\mathcal{R}(I^n)$ are Cohen–Macaulay for all $n \geq 1$.

The aim of this paper is to generalize **SdS-1** to schemes $X = \mathrm{Proj}(\mathcal{R}(I))$ with bounded projective Cohen–Macaulay deviation (Theorem 2.2).

In the first section we deal with the finiteness dimension $fg(\mathcal{R}(I))$ of the local cohomology groups of the Rees algebra [1, 4]. Under the hypothesis $fg(\mathcal{R}(I)) \geq r+1$ we characterize the condition $\mathrm{depth}(\mathrm{gr}_{I^n}(R)) \geq r$ in terms of the vanishing of the first r local cohomology groups of $\mathcal{R}(I)$ (Proposition 1.1). Sancho de Salas' exact sequence permits us to relate the vanishing of the local cohomology groups of the Rees algebra to the vanishing of the cohomology groups of $\mathrm{Proj}(\mathcal{R}(I))$ with supports in the exceptional divisor (Proposition 1.3).

In the second section we link the projective Cohen–Macaulay deviation to the finiteness dimension (Proposition 2.1). From the results of the first section we generalize **SdS-1** to schemes $\mathrm{Proj}(\mathcal{R}(I))$ with bounded projective Cohen–Macaulay deviation (Theorem 2.2). As a by-product we propose in Remark 2.3 an explanation of Cutkosky's and Huckaba–Huneke's examples; in particular these examples show that Theorem 2.2 is sharp. In Remark 2.4 we perform some explicit computations on Huckaba–Huneke's examples by means of CoCoa.

1. ON THE VANISHING OF LOCAL COHOMOLOGY GROUPS

Let $(R, \mathbf{m})$ be a Noetherian local ring and let I be an ideal of R. We denote by $\mathcal{R}(I) = \bigoplus_{n\geq 0} I^n t^n$ the Rees algebra of R with respect to I. We denote by $\mathcal{M} = \mathbf{m} \oplus It \oplus I^2t^2 \oplus \cdots$ the maximal homogeneous ideal of $\mathcal{R}(I)$ and by $\mathcal{R}(I)_+ = \oplus It \oplus I^2t^2 \oplus \cdots$ the irrelevant ideal of $\mathcal{R}(I)$. In this paper we already assume that $\dim(\mathcal{R}(I)) = d$ (see [9]). Given an ideal $J \subset R$, we denote by $\mathrm{gr}_J(R)$ the associated graded ring of R with respect to J, i.e., $\mathrm{gr}_J(R) = \bigoplus_{n\geq 0} J^n/J^{n+1}$.

We define $fg(\mathcal{R}(I))$ as the last integer $k \geq 0$ such that $H^i_{\mathcal{M}}(\mathcal{R}(I))_n = 0$ for all but finitely many n and for all $i < k$ (see [1, 9.1.3 and 13.1.17] and [4, 14]).

PROPOSITION 1.1. *Let $(R, \mathbf{m})$ be a d-dimensional Noetherian local ring and let I be an ideal of R. Let us assume $fg(\mathcal{R}(I)) \geq r+1$, $0 \leq r \leq d$; then the following conditions are equivalent*:

(i) *there exists a positive integer n_0 such that for all $n \geq n_0$ it holds that* $\operatorname{depth}(\operatorname{gr}_{I^n}(R)) \geq r$;

(ii) $H^i_{\mathcal{M}}(\mathcal{R}(I))_0 = 0$ *for* $i = 0, \ldots, r$;

(iii) *there exists a positive integer n_0 such that for all $n \geq n_0$ it holds that* $\operatorname{depth}(\mathcal{R}(I^n)) \geq r+1$.

Proof. Let us assume that (i) holds. From [9, Proposition 3.6] we get $\operatorname{depth}(\mathcal{R}(I^n)) \geq r$ for all $n \geq n_0$. Let $\mathcal{R}(I^n)$ be the nth Veronese transform of $\mathcal{R}(I)$ [1, 12.4.6]. Let $\mathcal{M}^{(n)} = \mathbf{m} \oplus I^n t \oplus I^{2n}t^2 \oplus \cdots$ be the maximal homogeneous ideal of $\mathcal{R}(I^n)$. Hence $H^i_{\mathcal{M}^{(n)}}(\mathcal{R}(I^n))_s = H^i_{\mathcal{M}}(\mathcal{R}(I))_{ns}$ for all $s \in \mathbb{Z}$, $i = 0, \ldots, r$. In particular we get $H^i_{\mathcal{M}}(\mathcal{R}(I))_0 = 0$ for $i = 0, \ldots, r-1$.

Let us consider the Huneke sequence of $\mathcal{R}(I^n)$-graded modules

$$0 \longrightarrow \mathcal{R}(I^n)(1) \longrightarrow \mathcal{R}(I^n) \longrightarrow \operatorname{gr}_{I^n}(R) \longrightarrow 0.$$

From the long exact sequence of local cohomology with respect to $\mathcal{M}^{(n)}$ we get in degree s the exact sequence of R-modules

$$\cdots \longrightarrow H^{r-1}_{\mathcal{M}^{(n)}}(\operatorname{gr}_{I^n}(R))_s \longrightarrow H^r_{\mathcal{M}^{(n)}}(\mathcal{R}(I^n))_{s+1} \longrightarrow H^r_{\mathcal{M}^{(n)}}(\mathcal{R}(I^n))_s \longrightarrow \cdots.$$

Notice that by (i) the left-hand-side module is zero. On the other hand, from the assumption $fg(\mathcal{R}(I)) \geq r+1$ we get that the right-hand-side module is zero for $s \ll 0$. By induction on s we get that $H^r_{\mathcal{M}^{(n)}}(\mathcal{R}(I^n)) = 0$; in particular we have $H^r_{\mathcal{M}^{(n)}}(\mathcal{R}(I^n))_0 = H^r_{\mathcal{M}}(\mathcal{R}(I))_0 = 0$.

Let us assume that (ii) holds. From the assumption $fg(\mathcal{R}(I)) \geq r+1$ and Serre's finiteness theorem [1, 20.4.6], we get that there exists an integer n_0 such that $H^i_{\mathcal{M}}(\mathcal{R}(I))_{ns} = 0$ for all $s \in \mathbb{Z}\backslash\{0\}$, $n \geq n_0$, and $i = 0, \ldots, r$. From (ii) and the fact $H^i_{\mathcal{M}^{(n)}}(\mathcal{R}(I^n))_* = H^i_{\mathcal{M}}(\mathcal{R}(I))_{n*}$ we get $H^i_{\mathcal{M}}(\mathcal{R}(I))_{ns} = 0$ for all $s \in \mathbb{Z}$, $n \geq n_0$, and $i = 0, \ldots, r$. Hence we get (iii).

Let us assume that $\operatorname{depth}(\mathcal{R}(I^n)) \geq r+1$. If $\operatorname{depth}(\operatorname{gr}_{I^n}(R)) < d$, then from [9, Proposition 3.6] we get $\operatorname{depth}(\operatorname{gr}_{I^n}(R)) = \operatorname{depth}(\mathcal{R}(I^n)) - 1 \geq r$. If $\operatorname{depth}(\operatorname{gr}_{I^n}(R)) \geq d$, then we also get (i). ∎

Remark 1.2. Notice that from the definition of $fg(\mathcal{R}(I))$ it is easy to see that if $\operatorname{depth}(\mathcal{R}(I^n)) \geq r+1$ then $fg(\mathcal{R}(I)) \geq r+1$, i.e., the condition $fg(\mathcal{R}(I)) \geq r+1$ is a necessary hypothesis in Proposition 1.1.

Let $(R, \mathbf{m})$ be a d-dimensional Noetherian local ring and let I be an ideal of R. We denote by $\pi: X = \operatorname{Proj}(\mathcal{R}(I)) \to \operatorname{Spec}(R)$ the blowing up of $\operatorname{Spec}(R)$ with respect to the closed sub-scheme defined by I. We denote by E the closed fiber of π, i.e., $E = X \otimes_R \mathbf{k}$.

We recall Karen Smith's version of the so-called Sancho de Salas sequence (see [12, p. 150]. For all $n \in \mathbb{Z}$ and $i \geq 0$ there exists a exact sequence of graded $\mathcal{R}(I)$-modules,

$$\text{(SdS)} \qquad \cdots \longrightarrow H^i_{\mathcal{M}}(\mathcal{R}(I)) \longrightarrow \bigoplus_{n\in\mathbb{Z}} H^i_{\mathbf{m}}(I^n) \longrightarrow \bigoplus_{n\in\mathbb{Z}} H^i_E(X, \mathcal{O}_X(n)) \longrightarrow H^{i+1}_{\mathcal{M}}(\mathcal{R}(I)) \longrightarrow \cdots.$$

The next proposition links condition (ii) of Proposition 1.1 with the cohomology groups $H^*_E(X, \mathcal{O}_X)$ under the assumption that R is Cohen–Macaulay.

PROPOSITION 1.3. *Let $(R, \mathbf{m})$ be a d-dimensional Cohen–Macaulay local ring and let I be an ideal of R. For all $0 \leq r \leq d-1$ the following conditions are equivalent*:

(ii) $H^i_{\mathcal{M}}(\mathcal{R}(I))_0 = 0$ *for all* $i = 0, \ldots, r$;

(iv) $H^i_E(X, \mathcal{O}_X) = 0$ *for all* $i = 0, \ldots, r-1$.

If the natural morphism $H^d_{\mathbf{m}}(R) \to H^d_E(X, \mathcal{O}_X)$ is a monomorphism then the above conditions are equivalent for $r = d$.

Proof. Let us consider an (SdS) sequence in degree $n = 0$ and $i = 0, \ldots, r$,

$$\cdots \longrightarrow H^{i-1}_{\mathbf{m}}(R) \longrightarrow H^{i-1}_E(X, \mathcal{O}_X) \longrightarrow H^i_{\mathcal{M}}(\mathcal{R}(I))_0 \longrightarrow H^i_{\mathbf{m}}(R) \longrightarrow \cdots.$$

Since R is Cohen–Macaulay, (ii) implies (iv), and (iv) implies (ii) if $r < d$.

Let us assume that (ii) holds and $r = d$. A similar argument for the previous case shows that $H^i_{\mathcal{M}}(\mathcal{R}(I))_0 = 0$ for $i = 0, \ldots, d-1$, so we have to prove that $H^d_{\mathcal{M}}(\mathcal{R}(I))_0 = 0$. The (SdS) sequence gives

$$\cdots \longrightarrow 0 = H^{d-1}_E(X, \mathcal{O}_X) \longrightarrow H^d_{\mathbf{m}}(\mathcal{R}(I))_0 \longrightarrow H^d_{\mathbf{m}}(R) \longrightarrow H^d_E(X, \mathcal{O}_X) \longrightarrow \cdots,$$

then the claim follows from the injectivity of $H^d_{\mathbf{m}}(R) \to H^d_E(X, \mathcal{O}_X)$. ∎

Remark 1.4. Notice that if R is pseudo-rational then $H^d_{\mathbf{m}}(R) \to H^d_E(X, \mathcal{O}_X)$ is a monomorphism ([13, Sect. 2] and [11]). On the other hand, this morphism is always surjective (see [13, Remark **b**, p. 103]).

2. ON THE SANCHO DE SALAS RESULT

In this section we generalize Sancho de Salas' result **SdS-1** to any value of $\operatorname{depth}(\operatorname{gr}_I(R))$, and we perform some explicit computations on Huckaba–Huneke's examples.

Let $(R, \mathbf{m})$ be a d-dimensional Noetherian local ring and let I be an ideal of R. If $X = \operatorname{Proj}(\mathcal{R}(I))$ then we denote by $\operatorname{pcmd}(X)$ the projective Cohen–Macaulay deviation of X, i.e., the maximum of $\dim(\mathcal{O}_{(X,x)}) - \operatorname{depth}(\mathcal{O}_{(X,x)})$, where $x \in X$.

We denote by $\operatorname{gdepth}(\mathcal{R}(I))$ the so-called generalized depth of $\mathcal{R}(I)$ with respect to the irrelevant ideal $\mathcal{R}(I)_+$ (see [10]). $\operatorname{gdepth}(\mathcal{R}(I))$ is the greatest integer $k \geq 0$ such that $\mathcal{R}_+ \subset \operatorname{rad}(\operatorname{ann}_{\mathcal{R}} H^i_{\mathcal{M}}(\mathcal{R}(I)))$ for all $i < k$. From the proof of [16, Lemma 2.2], we get that $\operatorname{gdepth}(\mathcal{R}(I)) \geq r$ if and only if there exists a positive integer v such that $H^i_{\mathcal{M}}(\mathcal{R}(I))_n = 0$ for $n \leq -v$ and $i = 0, \ldots, r-1$. In other words, $fg(\mathcal{R}(I)) = \operatorname{gdepth}(\mathcal{R}(I))$ (see also [14, Proposition 2.3]).

PROPOSITION 2.1. *Let $(R, \mathbf{m})$ be a d-dimensional Noetherian local ring which is the homomorphism image of a quotient of a Gorenstein ring. Let I be an ideal of R if $X = \operatorname{Proj}(\mathcal{R}(I))$; then*

$$(d+1) - \operatorname{pcmd}(X) = fg(\mathcal{R}(I)).$$

Proof. From [9, Lemma 2.2], we have that $\operatorname{gdepth}(\mathcal{R}(I))$ is the least integer

$$\operatorname{depth}(\mathcal{R}(I)_P) + \dim\left(\frac{\mathcal{R}(I)}{P}\right)$$

for $P \in \operatorname{Proj}(\mathcal{R}(I))$. Since R is universally catenary we get that $\mathcal{R}(I)$ is catenary. Then we have

$$\begin{aligned} \dim\left(\frac{\mathcal{R}(I)}{P}\right) &= ht\left(\frac{\mathcal{M}}{P}\right) \\ &= ht(\mathcal{M}) - ht(P) \\ &= (d+1) - \dim(\mathcal{R}(I)_P), \end{aligned}$$

so $fg(\mathcal{R}(I)) = \operatorname{gdepth}(\mathcal{R}(I))$ is the least integer

$$d + 1 - (\dim(\mathcal{R}(I)_P) - \operatorname{depth}(\mathcal{R}(I)_P))$$

$P \in \operatorname{Proj}(\mathcal{R}(I))$. From [7, Corollary 12.18], we get the claim. ∎

The next theorem is the main result of this paper.

THEOREM 2.2. *Let* $(R, \mathbf{m})$ *be a d-dimensional Noetherian local ring which is the homomorphic image of a quotient of a Gorenstein ring. Let* I *be an ideal of* R; *we set* $X = \mathrm{Proj}(\mathcal{R}(I))$. *If* $\mathrm{pcmd}(X) \leq d - r$, $0 \leq r \leq d$, *then the following conditions are equivalent*:

(i) *there exists a positive integer* n_0 *such that for all* $n \geq n_0$ *it holds that* $\mathrm{depth}(\mathrm{gr}_{I^n}(R)) \geq r$;

(ii) $H^i_{\mathcal{M}}(\mathcal{R}(I))_0 = 0$ *for* $i = 0, \ldots, r$;

(iii) *there exists a positive integer* n_0 *such that for all* $n \geq n_0$ *it holds that* $\mathrm{depth}(\mathcal{R}(I^n)) \geq r + 1$;

(iv) $H^i_E(X, \mathcal{O}_X) = 0$ *for all* $i = 0, \ldots, r - 1$.

Proof. Let us assume $r \leq d - 1$. From Proposition 2.1 we get that the condition $\mathrm{pcmd}(X) \leq d - r$ is equivalent to $fg(\mathcal{R}(I)) \geq r + 1$. From Proposition 1.1 and Proposition 1.3 we deduce that the four conditions are equivalent.

If $r = d$ then $fg(\mathcal{R}(I)) = d + 1$ (Proposition 2.1). Since X is Cohen–Macaulay from Proposition 1.1, Proposition 1.3, and [12, Theorem 4.3] we get the claim. ∎

Remark 2.3. The last result provides an explanation of Cutkosky's and Huckaba–Huneke's examples. In both cases R is a three-dimensional local ring homomorphic image of a Gorenstein ring. I is an $\mathbf{m}$-primary ideal such that $X = \mathrm{Proj}(\mathcal{R}(I))$ is a normal scheme and $\mathrm{gr}_{I^n}(R)$ is not Cohen–Macaulay for $n \gg 0$. Since $X = \mathrm{Proj}(\mathcal{R}(I))$ a normal non-Cohen–Macaulay scheme we have $\mathrm{pcmd}(X) = 1$, and the last result gives equivalent conditions only for $\mathrm{depth}(\mathrm{gr}_{I^n}(R)) \geq 2$. In fact, we have $R^2\pi_*(\mathcal{O}_X) \neq 0$, so $H^3_{\mathcal{M}}(\mathcal{R}(I))_0 \neq 0$ and $\mathrm{depth}(\mathrm{gr}_{I^n}(R)) \leq 2$.

Notice that Cutkosky's and Huckaba–Huneke's examples show that Theorem 2.2 is sharp for $d = 3$ and $r = 2$.

Remark 2.4. We can show that $\mathrm{depth}(\mathrm{gr}_I(R)) = 2$ by means of CoCoa [2]. Let us consider Huckaba–Huneke's example in the characteristic zero case. We have $R = \mathbb{Q}[x, y, z]_{(x, y, z)}$, and let us consider the (x, y, z)-primary ideal $I = (x^4, x(y^3 + z^3), y(y^3 + z^3), z(y^3 + z^3)) + (x, y, z)^5$. With the help of CoCoa we can compute the Poincaré series P_I of I,

$$P_I = \frac{T^3 + T^2 + 43T + 31}{(1 - T)^3},$$

and the Poincaré series of $I/(x^4, y(y^3 + z^3))$,

$$P_{I/(x^4, y(y^3+z^3))} = \frac{T^3 + T^2 + 43T + 31}{(1 - T)}.$$

Since $P_{I/(x^4,\, y(y^3+z^3))} = (1-T)^2 P_I$ we get that the initial forms of x^4, $y(y^3 + z^3)$ in $\mathrm{gr}_I(R)$ are a regular sequence [15]. $\mathrm{gr}_I(R)$ is not Cohen–Macaulay (Remark 2.3), so that $\mathrm{depth}(\mathrm{gr}_I(R)) = 2$.

REFERENCES

1. M. P. Brodman and R. Y. Sharp, "Local Cohomology," Cambridge Studies in Advanced Mathematics, Vol. 60, Cambridge Univ. Press, Cambridge, UK, 1998.
2. A. Capani, G. Niesi, and L. Robbiano, CoCoA, a system for doing computations in commutative algebra, 2000, available via anonymous ftp from `cocoa.dima.unige.it`.
3. S. D. Cutkosky, A new characterization of rational surface singularities, *Invent. Math.* **102** (1990), 157–177.
4. G. Faltings, Über die Annulatoren Lokaler Kohomologiegruppen, *Arch. Math.* **30** (1978), 473–476.
5. H. Grauert and O. Riemenschneider, Verschwindungssätze für analytische kohomologiegruppen auf komplexen räumen, *Invent. Math.* **11** (1970), 263–292.
6. R. Hartshorne and A. Ogus, On the factoriality of local rings of small embedding codimension, *Comm. Algebra* **1**, No. 5 (1974), 415–437.
7. M. Herrmann, S. Ikeda, and U. Orbanz, "Equimultiplicity and Blowing Up," Springer-Verlag, New York, 1980.
8. S. Huckaba and C. Huneke, Normal ideals in regular rings, *J. Reine Angew. Math.* **510** (1999), 63–82.
9. S. Huckaba and T. Marley, Depth formulas for certain graded rings associated to an ideal, *Nagoya Math. J.* **133** (1994), 57–69.
10. S. Huckaba and T. Marley, On associated graded rings of normal ideals, preprint, 2000.
11. C. Heneke, "Tight Closure and Its Applications," Conference Board of the Mathematical Sciences, Washington, DC, 1996.
12. J. Lipman, Cohen–Macaulayness in graded rings, *Math. Res. Lett.* **1** (1994), 149–157.
13. J. Lipman and B. Teissier, Pseudo-rational local rings and a theorem of Briançon–Skoda about integral closures of ideals, *Michigan Math. J.* **28** (1981), 97–116.
14. T. Marley, Finitely graded local cohomology and the depths of graded algebras, *Proc. Amer. Math. Soc.* **123**, No. 12 (1995), 3601–3607.
15. B. Singh, Effect of a permisible blowing-up on the local Hilbert function, *Invent. Math.* **26** (1974), 201–212.
16. N. V. Trung and S. Ikeda, When is the Rees algebra Cohen–Macaulay? *Comm. Algebra* **17** (1989), 2893–2922.
17. G. Valla, Certain graded algebras are Cohen–Macaulay, *J. Algebra* **42** (1976), 537–548.